

Quantum coding in non-inertial frames

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Abstract. The capacity of accelerated channel is investigated for different classes of initial states. It is shown that, the capacities of the travelling channels depend on the frame in which the accelerated channels are observed in and the initial shared state between the partners. In some frames, the capacities decay as the accelerations of both qubit increase. The decay rate is larger if the partners are initially share a maximum entangled state. The possibility of using the accelerated quantum channels to perform quantum coding protocol is discussed. The amount of decoded information is quantified for different cases, where it decays as the partner's accelerations increase to reach its minimum bound. This minimum bound depends on the initial shared states and it is large for maximum entangled state.

1. Introduction

Nowadays, investigating the dynamics of quantum states in non inertial frames represents one of the most events in the context of quantum information. For example, M. del Rey et al.[1] have presented a scheme for simulating a set of accelerated atoms coupled to a single mode field. The dynamics of multipartite entanglement of fermionic systems in non-inertial frames is discussed by Wang and Jing [2]. The effect of decoherence on a qubit-qutrit system is studied by Ramzan and Khan [3]. The influence of Unruh effect on the payoff function of the players in the quantum Prisoners is investigated in [4]. Goudarzi and Beyrami [5] have discussed the effect of uniform acceleration on multiplayer quantum game. The usefulness classes of travelling entangled quantum channels in non-inertial frames is investigated by Metwally [6].

Manipulating some quantum information tasks in these non-inertial frames represents a real desired aim. Therefore some efforts have been done to investigate quantum teleportation via accelerated states as quantum channels. For example, Alsing and Milburn [7] gave a description of the quantum teleportation between two users, one of them in an inertial frame and the other moves in an accelerated frame. In [8], the possibility of using maximum and partial entangled states as an initial states to perform quantum teleportation in non-inertial frames is discussed, where it is assumed that the teleported information can be either accelerated or non accelerated.

Quantum coding protocols represent one of the most important ways to send coded information between two users[9, 10, 11]. Therefore, it is important to discuss the possibility of using different accelerated states, which are initially prepared in maximum or partial entangled states to perform quantum coding protocol. In this contribution, we investigate the behavior of the capacities of different accelerated states. These states have been used as quantum channels to implement the original quantum coding protocol[9].

The paper is organized as follows: In Sec.2, the initial system is described, where we assume that the users share a two- qubit state of X-type [12]. The relation between Minkowski and Rindler spaces is reviewed. The Unruh effect on the initial system is investigated. The channels' capacities are quantified for different classes of initial states in Sec.3. The accelerated states are used as quantum channels to perform quantum coding. The amount of decoded information is calculated and the effect of the accelerations and the initial states setting is discussed in Sec.4. Finally, Sec.5 is devoted to discuss our results.

2. The Model

The class of X -state [12] represents one of most popular classes of quantum channel which have been extensively studied in the context of quantum information. Some properties of this class have been discussed in different directions. For example the phenomena of quantum discord of two-qubit X -states is discussed by Q. Chen et. al,

[13, 14]. The phenomena of entanglement sudden death of two-qubit X-states in thermal reservoirs is discussed in [15]. Therefore we are motivated to investigate the behavior of this state in non-inertial frame[6]. In computational basis $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$, X-state takes the form,

$$\begin{aligned} \rho_X = & \mathcal{A}_{11}|00\rangle\langle 00| + \mathcal{A}_{22}|01\rangle\langle 01| + \mathcal{A}_{33}|10\rangle\langle 10| + \mathcal{A}_{44}|11\rangle\langle 11| + \mathcal{A}_{23}|01\rangle\langle 10| \\ & + \mathcal{A}_{32}|10\rangle\langle 01| + \mathcal{A}_{14}|00\rangle\langle 11| + \mathcal{A}_{41}|11\rangle\langle 00|, \end{aligned} \quad (1)$$

where

$$\begin{aligned} \mathcal{A}_{11} = \mathcal{A}_{44} = & \frac{1}{4}(1 + c_z), \quad \mathcal{A}_{22} = \mathcal{A}_{33} = \frac{1}{4}(1 - c_z), \\ \mathcal{A}_{23} = \mathcal{A}_{32} = & \frac{1}{4}(c_x + c_y), \quad \mathcal{A}_{14} = \mathcal{A}_{41} = \frac{1}{4}(c_x - c_y). \end{aligned} \quad (2)$$

Since, we are interested to use this class of states to perform quantum coding in non-inertial frames, it is important to shed some light on the Minkowski and Rindler spaces in the context of Dirac field.

2.1. Minkowski and Rindler's spaces

In the inertial frames, Minkowski coordinates (t, z) are used to describe Dirac field, while in the uniformly accelerated case, Rindler coordinates (τ, χ) are more adequately. The relations between the Minkowski and Rindler coordinates are given by[7, 16],

$$\tau = r \tanh\left(\frac{t}{z}\right), \quad \chi = \sqrt{t^2 - z^2}, \quad (3)$$

where $-\infty < \tau < \infty$, $-\infty < \chi < \infty$ and r is the acceleration of the moving particle. The transformations (3) define two regions in Rindler's spaces: the first region I for $|t| < x$ and the second region II for $x < -|t|$. A single mode k of fermions and anti-fermions in Minkowski space is described by the annihilation operators a_k and b_{-k} respectively, where $a_k|0_k\rangle = 0$ and $b_{-k}^\dagger|0_k\rangle = 0$. In terms of Rindler's operator($c_k^{(I)}, d_{-k}^{(II)}$), the Minkowski operators can be written as [17],

$$a_k = \cos r c_k^{(I)} - \exp(-i\phi) \sin r d_{-k}^{(II)}, \quad b_{-k}^\dagger = \exp(i\phi) \sin r c_k^{(I)} + \cos r d_{-k}^{(II)}, \quad (4)$$

where $\tan r = e^{-\pi\omega\frac{c}{a}}$, $0 \leq r \leq \pi/4$, a is the acceleration such that $0 \leq a \leq \infty$, ω is the frequency of the travelling qubits, c is the speed of light and ϕ is an unimportant phase that can be absorbed into the definition of the operators. It is clear that, the transformation (4) mixes a particle in region I and an anti particle in region II . This effect is called Unruh effect [19, 20]. In terms of Rindler's modes, the Minkowski vacuum $|0_k\rangle_M$ and the one particle states $|1_k\rangle_M$ take the form,

$$\begin{aligned} |0_k\rangle_M = & \cos r |0_k\rangle_I |0_{-k}\rangle_{II} + \sin r |1_k\rangle_I |1_{-k}\rangle_{II}, \\ |1_k\rangle_M = & a_k^\dagger |0_k\rangle_M = |1_k\rangle_I |0_{-k}\rangle_{II}. \end{aligned} \quad (5)$$

2.2. Unruh effect on X -state

Since the expressions of the vacuum and single particle states are obtained in Rindler basis(5), then we can investigate the dynamics of the suggested state (1) from the uniformly accelerated point of view. In this context, using Eq.(5&1), one can obtain the form of the X -state in the first and the second regions.

$$\begin{aligned} \rho_{xA_\ell B_\ell} = & \mathcal{B}_{11}^\ell |00\rangle\langle 00| + \mathcal{B}_{22}^\ell |01\rangle\langle 01| + \mathcal{B}_{33}^\ell |10\rangle\langle 10| + \mathcal{B}_{44}^\ell |11\rangle\langle 11| + \mathcal{B}_{23}^\ell |01\rangle\langle 10| \\ & + \mathcal{B}_{32}^\ell |10\rangle\langle 01| + \mathcal{B}_{14}^\ell |00\rangle\langle 11| + \mathcal{B}_{41}^\ell |11\rangle\langle 00|, \end{aligned} \quad (6)$$

where, $\ell = I, II$, for the first and second regions respectively. If the states of Alice and Bob in the second region II are traced out, then the accelerated channel in the first region is defined by the coefficients,

$$\begin{aligned} \mathcal{B}_{11}^{(I)} &= \mathcal{A}_{11} \cos^2 r_a \cos^2 r_b, & \mathcal{B}_{14}^{(I)} &= \mathcal{A}_{14} \cos r_a \cos r_b \\ \mathcal{B}_{22}^{(I)} &= \cos^2 r_a (\mathcal{A}_{11} \sin^2 r_b + \mathcal{A}_{22}), & \mathcal{B}_{23}^{(I)} &= \mathcal{A}_{23} \cos r_a \cos r_b \\ \mathcal{B}_{23}^{(I)} &= \mathcal{A}_{32} \cos r_a \cos r_b & \mathcal{B}_{33}^{(I)} &= \cos^2 r_b (\mathcal{A}_{11} \sin^2 r_a + \mathcal{A}_{33}) \\ \mathcal{B}_{41}^{(I)} &= \mathcal{A}_{41} \cos r_a \cos r_b, & \mathcal{B}_{44}^{(I)} &= \sin^2 r_a (\mathcal{A}_{11} \sin^2 r_b + \mathcal{A}_{22}) + \mathcal{A}_{33} \sin^2 r_b + \mathcal{A}_{44} \end{aligned} \quad (7)$$

On the other hand, if we trace out the states of Alice and Bob in the first region I , the accelerated channel between Alice and Bob in the second region II is defined by the coefficients,

$$\begin{aligned} \mathcal{B}_{11}^{(II)} &= (\mathcal{A}_{22} + \mathcal{A}_{11} \cos^2 r_b) \cos^2 r_a + \mathcal{A}_{33} \cos^2 r_b + \mathcal{A}_{44}, & \mathcal{B}_{14}^{(II)} &= \mathcal{A}_{41} \sin r_a \sin r_b \\ \mathcal{B}_{22}^{(II)} &= (\mathcal{A}_{33} + \cos^2 r_a) \sin^2 r_b, & \mathcal{B}_{23}^{(II)} &= \mathcal{A}_{32} \sin r_a \sin r_b, \\ \mathcal{B}_{23}^{(II)} &= \mathcal{A}_{23} \sin r_a \sin r_b & \mathcal{B}_{33}^{(II)} &= (\mathcal{A}_{22} + \mathcal{A}_{11} \cos^2 r_b) \sin^2 r_a, \\ \mathcal{B}_{41}^{(II)} &= \mathcal{A}_{14} \sin r_a \sin r_b, & \mathcal{B}_{44}^{(II)} &= \mathcal{A}_{11} \sin^2 r_a \sin^2 r_b. \end{aligned} \quad (8)$$

There are also two channels that could be investigated, $\rho_{A_I B_{II}}$ and $\rho_{A_{II} B_I}$ which represent the channel between Alice, Anti-Bob and Anti-Alice, Bob respectively. In this context, we consider only the channel between Alice in the first region I and Bob in the second region II . In this case the coefficient \mathcal{B}_{ij} are given by,

$$\begin{aligned} \mathcal{B}_{11} &= (\mathcal{A}_{22} + \mathcal{A}_{11} \cos^2 r_b) \cos^2 r_a, & \mathcal{B}_{14} &= \mathcal{A}_{23} \cos r_a \sin r_b, \\ \mathcal{B}_{22} &= \mathcal{A}_{11} \cos^2 r_a \sin^2 r_b, & \mathcal{B}_{23} &= \mathcal{A}_{14} \cos r_a \sin r_b, \\ \mathcal{B}_{23} &= \mathcal{A}_{41} \cos r_a \sin r_b & \mathcal{B}_{33} &= (\mathcal{A}_{22} + \mathcal{A}_{11} \cos^2 r_b) \sin^2 r_a + (\mathcal{A}_{33} \cos^2 r_b + \mathcal{A}_{44}), \\ \mathcal{B}_{41} &= \mathcal{A}_{32} \cos r_a \sin r_b, & \mathcal{B}_{44} &= (\mathcal{A}_{33} + \mathcal{A}_{11} \sin^2 r_a) \sin^2 r_b. \end{aligned} \quad (9)$$

Since the quantum channels are obtained in the different regions, it is possible to investigate the behavior of the channel capacity as well as the possibility of using these states as quantum channels to perform quantum coding.

3. Channel capacity

Channel capacity which measures the rate of information transfer represents one of the most important indicators of the channel's efficiency. Therefore, it is necessary

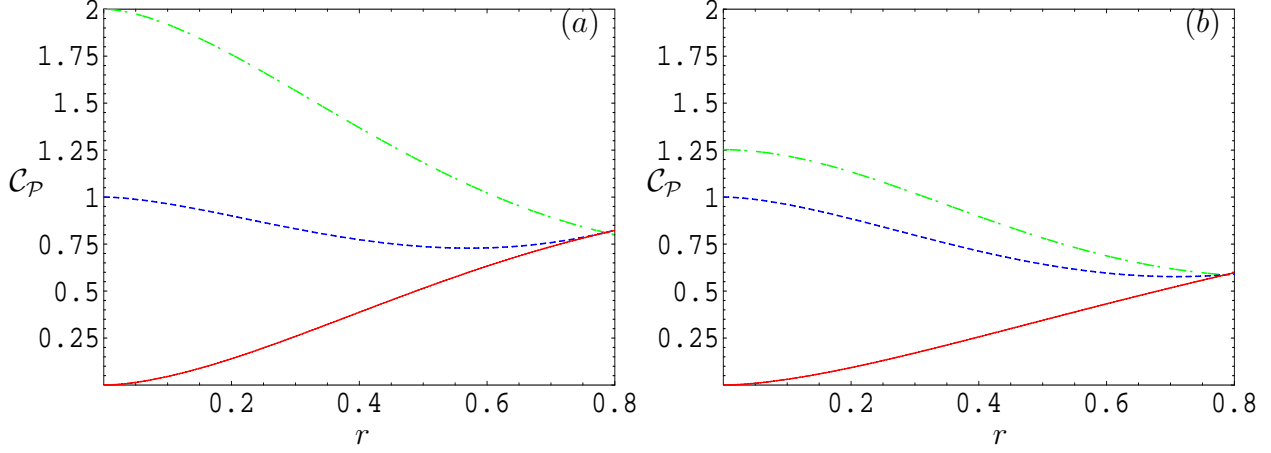


Figure 1. The channel's capacity of a system is initially prepared in (a) maximum entangled state, MES with $c_x = c_y = c_z = -1$ and (b) partial entangled state. PES with $c_x = -0.9, c_y = -0.8$ and $c_z = -0.7$. The dash dot, dot and solid curves represent the capacities of the channels $\rho_{A_I B_I}, \rho_{A_{II} B_{II}}$ and $\rho_{A_I B_{II}}$ respectively and $r_a = r_b = r$.

to investigate this phenomena for the travelling channels in different regions. Mathematically, the channel capacity of the channel $\rho_{A_\ell B_\ell}, \ell = I, II$ is given by [18],

$$\begin{aligned} \mathcal{C}_P = & \log_a D + (\mathcal{B}_{11}^{(\ell)} + \mathcal{B}_{33}^{(\ell)}) \log(\mathcal{B}_{11}^{(\ell)} + \mathcal{B}_{33}^{(\ell)}) + \mathcal{B}_{22}^{(\ell)} + \mathcal{B}_{44}^{(\ell)} \log(\mathcal{B}_{22}^{(\ell)} + \mathcal{B}_{44}^{(\ell)}) \\ & - \lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - \lambda_3 \log \lambda_1 - \lambda_3 \log \lambda_3 \end{aligned} \quad (10)$$

where,

$$\begin{aligned} \lambda_{1,2} = & \frac{1}{2} \left\{ \mathcal{B}_{11}^{(\ell)} + \mathcal{B}_{44}^{(\ell)} \right\} \pm \sqrt{(\mathcal{B}_{11}^{(\ell)} - \mathcal{B}_{44}^{(\ell)})^2 + 4\mathcal{B}_{41}^{(\ell)} \mathcal{B}_{14}^{(\ell)}}, \\ \lambda_{3,4} = & \frac{1}{2} \left\{ \mathcal{B}_{22}^{(\ell)} + \mathcal{B}_{33}^{(\ell)} \right\} \pm \sqrt{(\mathcal{B}_{22}^{(\ell)} - \mathcal{B}_{33}^{(\ell)})^2 + 4\mathcal{B}_{23}^{(\ell)} \mathcal{B}_{32}^{(\ell)}}, \end{aligned} \quad (11)$$

where $D = 2$. The channel capacity of the quantum channel between Alice in the first region and Bob in the second region is given by(10), but $\mathcal{B}_{ij}^{(\ell)}$ are given by (9).

The dynamics of the channel capacity of the travelling state is displayed in Fig.(1) for a system initially prepared in maximum entangled state. Fig.(1a), describes the behavior of \mathcal{C}_P for quantum channels which are generated between Alice and Bob in the first region, $\rho_{A_I B_I}$, in the second region $\rho_{A_{II} B_{II}}$ and between Alice in the first region and Bob in the second region $\rho_{A_I B_{II}}$. In this investigation, it is assumed that both qubits are accelerated. It is clear that the capacity the channel capacity \mathcal{C}_P is maximum for the channel $\rho_{A_I B_I}$ at zero accelerations. However as the accelerations of the travelling qubits increase, the channel capacity decreases smoothly to reach its minimum values with- ∞ acceleration. In the second region, the degree of entanglement of the state $\rho_{A_{II} B_{II}}$ is not maximum [21], therefore \mathcal{C}_P is not maximum at zero accelerations. For larger values of the accelerations the channel's capacity slightly decreases. This show that this generated entangled state is more robust than that in the first region, $\rho_{A_I B_I}$. The generated channel between Alice and Anti-Bob, $\rho_{A_I B_{II}}$ has a zero capacity at $r_a = r_b = 0$, since there is no entanglement generated between these qubits. However as the accelerations

increase, the channel's capacity $\mathcal{C}_{\mathcal{P}}$ increases gradually to reach its maximum value at $-\infty$ accelerations.

In Fig.(1b), the channel's capacities of the travelling quantum channels are investigated for a system initially prepared in a partial entangled state, where we set $c_x = -0.9, c_y = -0.8$ and $c_z = -0.7$. It is clear that the general behavior is similar to that predicted in Fig.(1a), for maximum entangled state, MES. However in the first region, the initial capacities of the state $\rho_{A_I B_I}$ is smaller than that described in Fig.(1a) for MES. As the accelerations of Alice and Bob's qubit increase the channel capacity $\mathcal{C}_{\mathcal{P}}$ decreases. In the second region II , the capacity of the quantum channel which is generated between Anti-Alice and Anti-Bob, $\rho_{A_{II} B_{II}}$ decreases smoothly as $r_{a(b)}$ increases. The capacity for the channel $\rho_{A_I B_{II}}$ increases as the accelerations of both qubit increase, but its maximum value is smaller than its corresponding one for MES.

From Fig.(1), one concludes that the channel capacity depends on the initial degree of entanglement of the travelling qubit. In our recent work [6], we have shown that the generated entangled channel in the first region has larger degree of entanglement than that in the second region. This explains, why the channel capacity in the first region is much larger than that displayed in the second region. On the other hand, initially, the channel between a qubit and the Anti-qubit of the other qubit is separable. However this channel turns into entangled state as the accelerations increase and consequently its capacity increases.

4. quantum Coding

In this section, we investigate the possibility of using the travelling channels between the different users to perform the quantum coding protocol which is proposed by Bennett and Wiener [9]. This protocol works as follows:

- (i) Alice encodes the given information in her qubit by using one of local unitary operators $\mathcal{U}_i = I, \sigma_x, \sigma_y$ or σ_z with probability p_i . Due to this operation the information is coded in the state

$$\rho_{cod} = \sum_i^3 \left\{ p_i \mathcal{U}_i \otimes I_2 \rho_{x_{A_\ell} B_\ell} I_2 \otimes \mathcal{U}_i^\dagger \right\}, \quad (12)$$

where I_2 is the unitary operator for Bob qubit.

- (ii) Alice sends her qubit to Bob, who makes joint measurements on the two qubits. The maximum amount of information which Bob can extract from Alices message is Bounded by [21, 22],

$$I_B = \mathcal{S}(\rho_{cod}) - \sum_{i=1}^3 p_i \mathcal{S}(\mathcal{U}_i \otimes I_2 \rho_{x_{A_\ell} B_\ell} I_2 \otimes \mathcal{U}_i^\dagger). \quad (13)$$

Explicitly,

$$I_B = -2(\lambda_+ \log \lambda_+ + \lambda_- \log \lambda_-) - 2(\lambda_1 \log \lambda_1 + \lambda_2 \log \lambda_2 + \lambda_3 \log \lambda_3 + \lambda_4 \log \lambda_4), \quad (14)$$

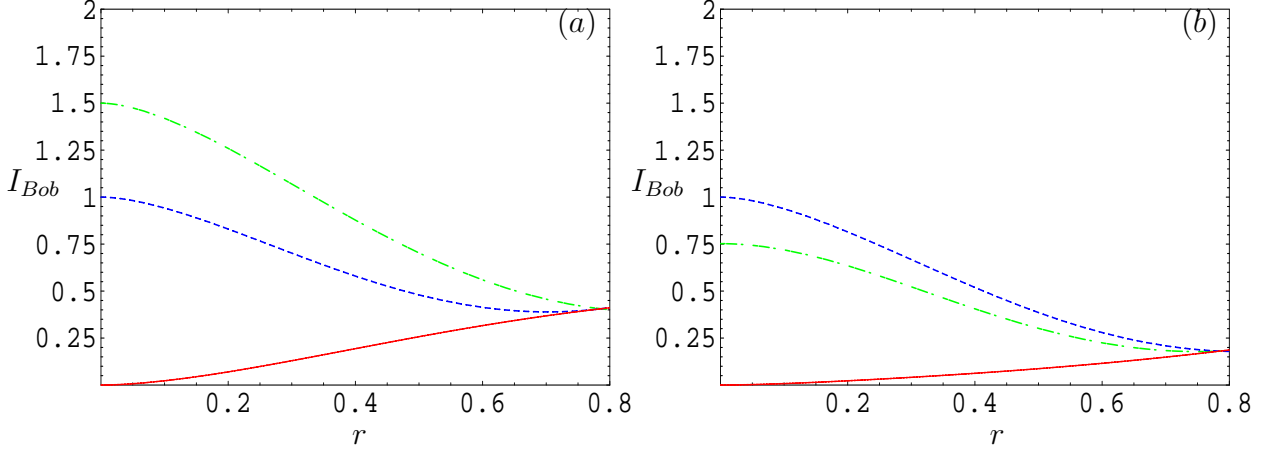


Figure 2. The amount of information decoded by Bob I_{Bob} for (a) MES and (b) PES, where we set the same parameters as described in Fig.(1).

where

$$\lambda_{\pm} = \frac{1}{2} \left\{ 1 + \frac{1}{2} \sqrt{\{(\mathcal{B}_{11}^{(\ell)} + \mathcal{B}_{33}^{(\ell)}) - (\mathcal{B}_{22}^{(\ell)} + \mathcal{B}_{44}^{(\ell)})\}^2 + 4(\mathcal{B}_{12}^{(\ell)} + \mathcal{B}_{34}^{(\ell)})(\mathcal{B}_{21}^{(\ell)} + \mathcal{B}_{43}^{(\ell)})} \right\}, \quad (15)$$

and $\lambda_i, i = 1..4$ are given in (11) and it is assumed that Alice has used the unitary operator with an equal probability i.e., $p_i = \frac{1}{4}$, $i = I, \sigma_x, \sigma_y, \sigma_z$.

In Fig.(2), the amount of decoded information by Bob, I_{Bob} is plotted for different initial channels. Fig.(2a), describes the behavior of I_{Bob} for a qubit system initially prepared in maximum entangled state. It is clear that the amount of information which decoded from the channel between Alice and Bob in the first region, $\rho_{A_I B_I}$ decreases smoothly to reach its minimum value when the accelerations tend to infinity. In the second region the information is coded in the state $\rho_{A_{II} B_{II}}$. In this case the amount of decoded information is smaller than that shown in the first region. However the decoded information from the channel between Alice and Anti-Bob, $\rho_{A_I B_{II}}$ increases as the accelerations increase to reach its maximum bound. This maximum bound represents the lower bound for the decoded information in the regions I and II .

The behavior of the decoded information in a system initially prepared in a partial entangled state is depicted in Fig.(2b). The general behavior is similar to that shown in Fig.(2a), but the information decoded by Bob, I_{Bob} is much smaller than coded in maximum entangled states. The minimum bound is smaller than that displayed for a system initially prepared in MES. The amount of information which is decoded from the quantum channel $\rho_{A_I B_{II}}$ increases, with a rate smaller than its corresponding one in Fig.(2a), as the accelerations increase.

5. Conclusion

In this contribution, we investigate the dynamics of the accelerated channels in non-inertial frame. It is assumed that the partners initially share maximum or partial entangled channels. The capacity of the accelerated channels depends on the initial state setting and the frame in which the partners are observed. It is shown that the capacity decays if both partners are observed in the same frame, and increases smoothly if the partners are observed in different frames. The capacity decays quickly in one region, where the accelerated channel has larger degree of entanglement. However in some regions, the decay is small because this accelerated channel is partially entangled. This shows that the capacity of larger degree of entanglement decays faster than that has smaller degree of entanglement. Starting from a maximum entangled states, the capacity of the accelerated channels is much larger than that depicted for partial entangled channel as initial states.

The possibility of employing the accelerated channels in different frames to perform quantum coding is investigated. It is assumed that the source supplies the partners with different classes of initial states. The amount of decoded information is quantified for different accelerated channels. It is shown the decoded information decays as the the accelerations of both qubit increases. The decay rate depends on the frame in which the channel is accelerated, where in the first region, the rate of decay is larger than that depicted in the second region. If the partners start with a maximum entangled states, then the decoded information from the accelerated channels is much larger than that decoded from accelerated channel generated initially from partial entangled channels. The decoded information from a channel between a partner and the Anti-partner increases gradually to reach its maximum bound with- ∞ acceleration. The increasing rate depends on the initially shared state between the partners.

In conclusion, it is possible to use the accelerated quantum channels to send coded information between two partners. The channel capacity and the amount of decoded information depend on the frames in which the partners are observed and the initial shared state between the partners.

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